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# Sequential criteria for the equality of uniform structures in $q$-groups

G. Hansel [a], J.P. Troallic *,[b]

[a] *URA CNRS D1378, Faculté des Sciences, Université de Rouen, F-76134 Mont-Saint-Aignan, France*
[b] *URA CNRS D1378, Faculté des Sciences et des Techniques, Université du Havre, 25, rue Philippe Lebon, F-76600 Le Havre, France*

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**Abstract**

Let $G$ be a $q$-group. We give simple new criteria for the equality of left and right uniform structures on $G$ to hold. These criteria imply several previously known results as immediate consequences.

*Key words:* $q$-groups; Left and right uniform structures

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## 1. Introduction

A locally compact topological group $G$ is said to be $\alpha$-compact if $\alpha$ is the least cardinal number such that $G$ can be written as a union of $\alpha$ compact subsets. In [3], Itzkowitz gave the following criterion for the equality of the left and right uniform structures on a locally compact and $\alpha$-compact group $G$ to hold: for every subset $B \subset G$ of cardinal less than or equal to $\alpha$ and every neighborhood $U$ of the identity $e$ in $G$, the set $\bigcap_{x \in B} xUx^{-1}$ is still a neighborhood of $e$. In the same work, Itzkowitz raised the possibility to discard cardinality considerations. And actually, Pestov [7] has shown by a transfinite argument that in the statement of Itzkowitz, it suffices to consider countable subsets. As a consequence, the left and right uniform structures on $G$ are equal iff the same is true of its open $\sigma$-compact subgroups.

* Corresponding author.


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In [2], making use of a theorem of Grothendieck [1], we have generalized Pestov's results and given the following sequential criterion for the equality of uniform structures on $G$: every "quasi-bounded" double sequence of the form $(g_n x_p g_n^{-1})$, with $g_n$, $x_p \in G$, has at least one double cluster point in $G$. It follows from this criterion that the uniform structures on $G$ are equal iff the same is true for every countable subgroup of $G$.

In [6], Milnes has tackled the same problem from a functional point of view: he has shown that the uniform structures on a locally compact group $G$ are equal iff the spaces of bounded left and right uniformly continuous functions on $G$ are the same.

In the present work, we give new and particularly simple criteria for the equality of left and right uniform structures. They are valid for the class of all $q$-groups (see definition inside) which is much more wider than the class of locally compact groups (in particular, recall that Čech complete groups are $q$-groups). These criteria are corollaries of Theorem 2.3 which is established in a purely elementary way and they imply all the results quoted above as immediate consequences.

## 2. Sequential criteria

Let $G$ be a topological group; we denote by $e$ its identity element and by $\mathscr{V}_e$ the set of all the neighborhoods of $e$.

**Lemma 2.1.** *Let $X$ be a symmetric neighborhood of $e$. Let $A$ be a subset of $G$ which is contained in the union of a finite family $(Xx_i)_{i\in I}$ of right translated sets of $X$. Then the set $\bigcap_{a\in A} a^{-1}X^3a$ is still a neighborhood of $e$.*

**Proof.** Since $X$ is symmetric, for all $a \in A$, there exists $i \in I$ such that $x_i \in Xa$ and this implies that $x_i^{-1}Xx_i \subset a^{-1}X^3a$. Hence $\bigcap_{i\in I} x_i^{-1}Xx_i \subset \bigcap_{a\in A} a^{-1}X^3a$ and therefore $\bigcap_{a\in A} a^{-1}X^3a$ is a neighborhood of $e$. □

*A neighborhood basis* of $e$ is a subset $\mathscr{W}$ of $\mathscr{V}_e$ such that for all $V \in \mathscr{V}_e$ there is $W \in \mathscr{W}$ contained in $V$. We denote by $\mathscr{B}$ the set of all the neighborhood bases of $e$.

The next lemma gives an essential tool used in the proof of Theorem 2.3.

**Lemma 2.2.** *Let $\mathscr{W}$ be a neighborhood basis of $e$, $X$ a symmetric neighborhood of $e$, $F$ a finite subset of $G$, $(x_W)_{W\in\mathscr{W}}$ and $(y_W)_{W\in\mathscr{W}}$ two families of elements of $G$ indexed by $\mathscr{W}$. Suppose that for all $W \in \mathscr{W}$, $x_W^{-1}y_W \in W$ and $x_W y_W^{-1} \notin X^3$. Then there exists $\mathscr{W}' \in \mathscr{B}$ such that $\mathscr{W}' \subset \mathscr{W}$ and such that for all $W' \in \mathscr{W}'$, $\{x_{W'}, y_{W'}\} \subset (XF)^c$.*

**Proof.** (1) Let

$$\mathscr{W}_1 = \{W \in \mathscr{W} \mid x_W \notin XF\}$$

and let us show that $\mathscr{W}_1$ is still a neighborhood basis of $e$. Let $V$ be a neighborhood of $e$. We have only to show that there is $W \in \mathscr{W}_1$ such that $W \subset V$. Let

$$\mathscr{W}_2 = \{W \in \mathscr{W} \mid W \subset V\}.$$

Since $\mathscr{W}$ is a neighborhood basis of $e$ and since $V$ is a neighborhood of $e$, $\mathscr{W}_2$ is a neighborhood basis of $e$.

Let us show that $\{x_W \mid W \in \mathscr{W}_2\}$ is not contained in the union of any finite subfamily of $(Xg)_{g \in G}$. Suppose the contrary; then by Lemma 2.1, the set $\bigcap_{W \in \mathscr{W}_2} x_W^{-1} X^3 x_W$ is a neighborhood of $e$. Let $Z \in \mathscr{W}_2$ be contained in this neighborhood; thus $x_Z^{-1} y_Z \in x_Z^{-1} X^3 x_Z$; hence $x_Z y_Z^{-1} \in X^3$, which is absurd.

Hence there exists $W \in \mathscr{W}_2$ such that $x_W \notin XF$. Therefore $W \in \mathscr{W}_1$ and since $W \subset V$, $\mathscr{W}_1$ is a neighborhood basis of $e$.

(2) Let

$$\mathscr{W}' = \{W \in \mathscr{W}_1 \mid y_W \notin XF\}.$$

By reasoning like in (1) with $\mathscr{W}_1$ instead of $\mathscr{W}$, we get that $\mathscr{W}'$ is a neighborhood basis of $e$ and that for all $W' \in \mathscr{W}'$, $\{x_{W'}, y_{W'}\} \subset (XF)^c$. $\square$

Let $G$ be a topological group. A sequence $(x_n)_{n \in \mathbb{N}}$ of points of $G$ is *right uniformly discrete* if there is a neighborhood $V$ of $e$ such that $Vx_p \cap Vx_q = \emptyset$ whenever $p \neq q$. It is equivalent to saying that there is a symmetric neighborhood $V'$ of $e$ such that $x_q \notin V'x_p$ whenever $q > p$.

A *q-group* is a topological group $G$ in which there exists a sequence $(U_n)_{n \in \mathbb{N}}$ of neighborhoods of $e$ with the following property: any sequence $(u_n)_{n \in \mathbb{N}}$ of points of $G$ such that $u_n \in U_n$, $n \in \mathbb{N}$, has a cluster point.

We now establish the main theorem of this paper.

**Theorem 2.3.** *Let $G$ be a q-group whose right and left uniform structures are different. Then there are two sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ of points of $G$ such that $e$ is a cluster point of the sequence $(x_n^{-1} y_n)_{n \in \mathbb{N}}$ and such that the mixed sequence $(x_0, y_0, x_1, y_1, \ldots, x_n, y_n, \ldots)$ is right uniformly discrete.*

**Proof.** Since the right and left uniform structures of $G$ are different, there is a symmetric neighborhood $V$ of $e$ such that $\bigcap_{x \in G} x^{-1} V x$ is not a neighborhood of $e$. Let $W$ be any symmetric neighborhood of $e$. Since $W$ is not contained in $\bigcap_{x \in G} x^{-1} V x$, there are $w \in W$ and $a \in G$ such that $w \notin a^{-1} V a$. Let $b = aw^{-1}$; then $a^{-1} b \in W$ and $ab^{-1} \notin V$. Hence there are two families $(a_W)_{W \in \mathscr{V}_e}$ and $(b_W)_{W \in \mathscr{V}_e}$ such that for all $W \in \mathscr{V}_e$,

$$a_W^{-1} b_W \in W \quad \text{and} \quad a_W b_W^{-1} \notin V.$$

Let $X$ be a symmetric neighborhood of $e$ such that $X^3 \subset V$ and let $Y$ be a closed symmetric neighborhood of $e$ such that $Y^2 \subset X$.

Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of neighborhoods of $e$ such that if $u_n \in U_n$ for all $n \in \mathbb{N}$, then the sequence $(u_n)_{n \in \mathbb{N}}$ has at least one cluster point. We can suppose that the $U_n$ are symmetric. We shall inductively build a sequence $(\mathscr{W}_n, Y_n, a_n, b_n)_{n\in\mathbb{N}}$ of elements of $\mathscr{B}\times\mathscr{V}_e\times G\times G$ in the following way.

We put $Y_0=Y$, $\mathscr{W}_0=\{W\in\mathscr{V}_e \mid W\subset U_0\cap Y_0\}$. Let $W$ be any element of $\mathscr{W}_0$; we put $a_0=a_W$ and $b_0=b_W$. Suppose that $(\mathscr{W}_i, Y_i, a_i, b_i)$, $i=0, 1,\ldots, n-1$, have been built. Let $Y_n=\bigcap_{r=0}^{n-1} b_r^{-1}Yb_r$ and let $\mathscr{W}'_{n-1}$ be the neighborhood basis of $e$ defined by $\mathscr{W}'_{n-1}=\{W\in\mathscr{W}_{n-1} \mid W\subset U_n\cap Y_n\}$. It follows from Lemma 2.2 that there is $\mathscr{W}_n\in\mathscr{B}$ such that $\mathscr{W}_n\subset\mathscr{W}'_{n-1}$ and such that for all $W\in\mathscr{W}_n$,

$$\{a_W, b_W\}\subset\left(X\{a_0, b_0,\ldots, a_{n-1}, b_{n-1}\}\right)^c.$$

Let us choose any element $W$ of $\mathscr{W}_n$; we put $a_n=a_W$ and $b_n=b_W$. Thus we have defined $(\mathscr{W}_n, Y_n, a_n, b_n)$.

The sequence $(a_n, b_n)$ of $G^2$ has the following properties: for all $n\in\mathbb{N}$,

$$a_n^{-1}b_n\in U_n\cap Y_n, \tag{1}$$

$$a_nb_n^{-1}\notin X^3, \tag{2}$$

$$\{a_n, b_n\}\subset\left(X\{a_0, b_0,\ldots, a_{n-1}, b_{n-1}\}\right)^c. \tag{3}$$

It follows from (1) and from the characteristic property of the sequence $(U_n)$ that the sequence $(a_n^{-1}b_n)_{n\in\mathbb{N}}$ has a cluster point $t$ or equivalently that $e$ is a cluster point of the sequence $(a_n^{-1}b_nt^{-1})_{n\in\mathbb{N}}$. Moreover for $m<n$, $a_n^{-1}b_n\in b_m^{-1}Yb_m$ and therefore, since $Y$ is closed, for all $m\in\mathbb{N}$,

$$t\in b_m^{-1}Yb_m. \tag{4}$$

Let for all $n\in\mathbb{N}$, $x_n=a_n$ and $y_n=b_nt^{-1}$, and let us show that the sequences $(x_n)$ and $(y_n)$ have the desired properties.

The sequence $(x_n^{-1}y_n)$ admits $e$ as a cluster point (since $x_n^{-1}y_n=a_n^{-1}b_nt^{-1}$). Let us show that the mixed sequence $(x_0, y_0,\ldots, x_n, y_n,\ldots)$ is right uniformly discrete. Since $Y$ is a symmetric neighborhood of $e$, it is sufficient to prove that for any two different positive integers $r$ and $n$, $x_n, y_n\subset(Y\{x_r, y_r\})^c$ and that for all $n\in\mathbb{N}$, $y_n\notin Yx_n$.

Let $r$ and $n$ be two different positive integers. It follows from (3) that $x_n\notin Xx_r$ and a fortiori that $x_n\notin Yx_r$. Similarly $y_n=b_nt^{-1}\notin Xb_rt^{-1}=Xy_r$ and a fortiori $y_n\notin Yy_r$.

Let us show that $x_n\notin Yy_r$, $r$ and $n$ being two positive integers (possibly equal). We can write

$$x_ny_r^{-1}=x_ntb_r^{-1}=\left(a_nb_r^{-1}\right)\left(b_rtb_r^{-1}\right).$$

It follows from (2) and (3) that $a_nb_r^{-1}\notin X$ and from (4) that $b_rtb_r^{-1}\in Y$. Since $Y^2\subset X$, we get that $x_ny_r^{-1}\notin Y$ and therefore $x_n\notin Yu_r$.

Thus the sequence $(x_0, y_0,\ldots, x_n, y_n,\ldots)$ is right uniformly discrete. □

Several simple criteria for the equality of left and right uniform structures on a *q-group* are corollaries of the previous theorem.

**Corollary 2.4.** *Let $G$ be a q-group. The following conditions are equivalent*:

(1) *The left and right uniform structures on $G$ are equal.*

(2) *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences of elements of $G$ such that $e$ is a cluster point of the sequence $(x_n^{-1}y_n)_{n\in\mathbb{N}}$; then $e$ is also a cluster point of the sequence $(x_ny_n^{-1})_{n\in\mathbb{N}}$.*

(3) *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences of elements of $G$ such that $e \in \overline{\{x_p^{-1}y_q \mid p, q\in\mathbb{N}\}}$; then $e\in\overline{\{x_py_q^{-1} \mid p, q\in\mathbb{N}\}}$.*

(4) *Any uniformly right discrete sequence of elements of $G$ is also uniformly left discrete.*

(5) *For any neighborhood $V$ of $e$ and any sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $G$, the set $\bigcap_{n\in\mathbb{N}} x_n^{-1}Vx_n$ is a neighborhood of $e$.*

**Proof.** Condition (1) implies conditions (2), (3), (4) and (5) because under condition (1) there exists a basis of symmetric neighborhoods of $e$ which are invariant under the inner automorphisms. Implications (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1), (4) $\Rightarrow$ (1), are immediate consequences of Theorem 2.3. To prove that (5) implies (1) let us show that (5) implies (2). Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences of elements of $G$ such that $e$ is a cluster point of the sequence $(x_n^{-1}y_n)_{n\in\mathbb{N}}$ and let $V$ be a neighborhood of $e$. It follows from the hypothesis that $\bigcap_{n\in\mathbb{N}} x_n^{-1}V^{-1}x_n$ is a neighborhood of $e$ and consequently $x_p^{-1}y_p$ belongs to $\bigcap_{n\in\mathbb{N}} x_n^{-1}V^{-1}x_n$ for an infinite number of indexes $p\in\mathbb{N}$; from this we deduce that $x_py_p^{-1}$ belongs to $V$ for an infinite number of indexes $p\in\mathbb{N}$. Hence $e$ is a cluster point of the sequence $(x_ny_n^{-1})_{n\in\mathbb{N}}$. □

The following result was already obtained in [2] by a much more involved method and only in the case of a locally compact group.

**Corollary 2.5.** *Let $G$ be a q-group. The following conditions are equivalent*:

(1) *The left and right uniform structures on $G$ are equal.*

(2) *The left and right uniform structures of every countable subgroup of $G$ are equal.*

**Proof.** (1) $\Rightarrow$ (2) is clear. Let us prove that (2) $\Rightarrow$ (1) by showing that condition (2) of Corollary 2.4 is satisfied. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences of elements of $G$ such that $e$ is a cluster point of the sequence $(x_n^{-1}y_n)_{n\in\mathbb{N}}$. Let $H$ be the topological subgroup of $G$ generated by the set $\{x_n \mid n\in\mathbb{N}\}\cup\{y_n \mid n\in\mathbb{N}\}$. Since $H$ is countable, the left and right uniform structures on $H$ are equal; hence $e$ is a cluster point in the topological space $H$ of the sequence $(x_ny_n^{-1})_{n\in\mathbb{N}}$ and consequently also in the topological space $G$. □

Let $G$ be a topological group. We denote by $\mathscr{U}_L(G)$ (respectively $\mathscr{U}_R(G)$) the space of all bounded complex valued functions on $G$ which are left (respectively right) uniformly continuous.

In [6], Milnes established the following result in the case of a locally compact group.

**Corollary 2.6.** *Let $G$ be a q-group. The left and right uniform structures on $G$ are equal if and only if $\mathscr{U}_L(G) = \mathscr{U}_R(G)$.*

**Proof.** Suppose that the left and right uniform structures on $G$ are not equal and let us show that $\mathscr{U}_L(G) \neq \mathscr{U}_R(G)$, the reciprocal statement being clearly true. It follows from Corollary 2.4 that there are two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of elements of $G$ such that $e \in \overline{\{x_p^{-1}y_q \mid p, q \in \mathbb{N}\}}$ and $e \notin \overline{\{x_p y_q^{-1} \mid p, q \in \mathbb{N}\}}$. Let $A = \{x_n \mid n \in \mathbb{N}\}$ and $B = \{y_n \mid n \in \mathbb{N}\}$; let $g : A \cup B \to \mathbb{C}$ be the function defined by $g(x) = 0$ if $x \in A$ and $g(x) = 1$ if $x \in B$. The function $g$ is right uniformly continuous; hence, by Katetov's theorem [4,5], it can be extended to a function $f \in \mathscr{U}_R(G)$. But since $g$ is not left uniformly continuous, $f \notin \mathscr{U}_L(G)$. □

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